The Saga of the matrix Grothendieck inequality

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Abstract

In 1953, Grothendieck proved a powerful inequality (GI) that he called "the fundamental theorem in the metric theory of tensor products". A matrix version (MGI) was given by Lindenstrauss-Pełczyński in 1968. It states that for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, there is a finite constant $K_G^{\mathbb{F}} > 0$ such that for every $d, m, n \in \mathbb{N}$ with $d \ge \min(m, n)$ and every matrix $A = (a_{ij}) \in \mathbb{F}^{m\times n}$,

$$\|A\|_{1,\infty}^{\mathbb{F}} = \max_{|\varepsilon_i|=|\delta_j|=1} \left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\bar{\varepsilon}_i\delta_j\right| \le$$

$$\|A\|_G^{\mathbb{F}} = \max_{\|x_i\|=\|y_j\|=1} \left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle x_i, y_j\rangle\right| \le K_G^{\mathbb{F}}\|A\|_{1,\infty}.$$

The maximum on the left is taken over all $\delta_i, \varepsilon_j \in \mathbb{F}$ of unit absolute value, $i = 1, \dots, m$, $j = 1, \dots, n$, and the maximum in the middle is taken over all $x_i, y_j \in \mathbb{F}^d$ of unit norm. The exact value of $K_G^{\mathbb{F}}$, which depends on the choice of $\mathbb{F}$, remains unknown but the best current upper and lower bounds are $1.67696 \le K_G^{\mathbb{R}} \le (\pi/2)\log(1+\sqrt{2}) \approx 1.78221$ and $1.33807 \le K_G^{\mathbb{C}} \le 1.40491$.

Grothendieck showed, using his inequality, that one can factor a bounded linear operator: $C \xrightarrow{\theta} L$ through a Hilbert space: $C \xrightarrow{\phi} H \xrightarrow{\psi} L$, such that $\|\phi\|\|\psi\| \le K_G^{\mathbb{F}}\|\theta\|$. The computational importance of MGI is that $\|A\|_{1,\infty}^{\mathbb{F}}$ is NP-hard to compute, while $\|A\|_G^{\mathbb{F}}$ is computable in polynomial time using semidefinite programming.

Our recent contributions are: First, [1], MGI for quaternions $\mathbb{H}$ and giving good lower and upper bounds on $K_G^{\mathbb{H}}$. Second, [2], symmetric MGI for $\mathbb{S}^n$: $n \times n$ symmetric matrices over $\mathbb{R}$, or Hermitian matrices over $\mathbb{C}$ or $\mathbb{H}$:

$$\|B\|_\theta^{\mathbb{F}} = \max_{|\delta_i|=1} \le \left|\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\delta_i^*\delta_j\right| \le \|B\|_\gamma^{\mathbb{F}} = \max_{\|x_i\|=1} \left|\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\langle x_i, x_j\rangle\right| \le K_\gamma^{\mathbb{F}}\|B\|_\theta,$$

$$\|B\|_\Theta := \max_{|\delta_i|\le 1} \left|\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\delta_i\bar{\delta}_j\right| \le \|B\|_\Gamma := \max_{\|x_i\|\le 1} \left|\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\langle x_i, x_j\rangle\right| \le K_\Gamma^{\mathbb{F}}\|B\|_\theta.$$

$$K_G^{\mathbb{F}} < K_\Gamma^{\mathbb{F}} < K_\gamma^{\mathbb{F}} \begin{cases} \le \sinh(\pi/2) \approx 2.30130 & \text{over } \mathbb{R}, \\ = 8/\pi - 1 \approx 1.54648 & \text{over } \mathbb{C}. \end{cases}$$

Furthermore in [3] we give an elementary and unified exposition of the proof of MGI, and in [4] we show that Grothendieck constant is norm of Strassen matrix multiplication tensor.

References

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